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Quasi-bi-Hamiltonian systems and separability

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Abstract. Two quasi-bi-Hamiltonian systems with three and four degrees of freedom are presented. These systems are shown to be separable in terms of Nijenhuis coordinates. Moreover, the most general Pfaffian quasi-bi-Hamiltonian system with an arbitrary number of degrees of freedom is constructed (in terms of Nijenhuis coordinates) and its separability is proved.

1. Preliminaries

As is known, the bi-Hamiltonian structure is a peculiar property of integrable systems, both finite and infinite dimensional [1, 2]. We recall some definitions. Let M be a differentiable manifold, TM and T^*M its tangent and cotangent bundle and P_0, $P_1 : T^*M \mapsto TM$ two compatible Poisson tensors on M [1]: a vector field, X, is said to be bi-Hamiltonian with respect to P_0 and P_1 if two smooth functions, H and F, exist such that

$$X = P_0\, \mathrm{d}H = P_1\, \mathrm{d}F \qquad (1.1)$$

with d denoting the exterior derivative. Moreover, if P_0 is invertible, the tensor $N := P_1 P_0^{-1}$ is a Nijenhuis (or hereditary) tensor; in terms of the gradients of the Hamiltonian functions, the bi-Hamiltonian property (1.1) entails that N^* (the adjoint map of N) maps iteratively $\mathrm{d}H$ into closed one-forms, so that $\mathrm{d}(N^{*^i}\mathrm{d}H) = 0$ $(i = 1, 2, \ldots)$.

As a matter of fact, it is in general quite difficult to directly construct a bi-Hamiltonian structure for a given integrable Hamiltonian vector field; so one can try to use some reduction procedure, starting from a few 'universal' Poisson structures defined in an extended phase space. On the other hand, in the case of finite-dimensional systems arising as restricted or stationary flows from soliton equations [3, 4], the final result of the reduction procedure is some physically interesting dynamical systems (for example the Hénon–Heiles system) which, in their natural phase space, satisfy a weaker condition than the bi-Hamiltonian one. So the notion of a quasi-bi-Hamiltonian (QBH) system can be introduced [5, 6]; it was applied in [7] to dynamical systems with two degrees of freedom. One of the aims of this paper is just to give explicit examples of QBH systems with more than two degrees of freedom.

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According to [7], a vector field, X, is said to be a QBH vector field with respect to two compatible Poisson tensors P_0 and P_1 if there are three smooth functions H, F, ρ such that

$$X = P_0\,\mathrm{d}H = \frac{1}{\rho} P_1\,\mathrm{d}F \tag{1.2}$$

(ρ playing the role of an integrating factor). From this equation it follows that F is an integral of motion for X, in involution with H, so that a QBH vector field with two degrees of freedom is Liouville-integrable. Of course, equation (1.2) can be studied for an arbitrary number, n, of degrees of freedom, but the knowledge of F and H is no longer sufficient to assure the integrability of X for $n > 2$. In this case, the search for integrability can be pursued using a sufficient criterion, which was recently introduced by one of the present authors (GT). Indeed, one can show the following proposition.

Proposition 1.1 ([4]). Let M be a $2n$-dimensional symplectic manifold equipped with an invertible Poisson tensor P_0, and let X be a Hamiltonian vector field with Hamiltonian H: $X = P_0\,\mathrm{d}H$. Let a tensor $N : TM \to TM$ exist such that the tensor $P_1 : T^*M \to TM$ defined by $P_1 := NP_0$ is skew-symmetric. Denote by $X_i := N^{i-1}X$ and $\alpha_i := N^{*^{i-1}}\,\mathrm{d}H$ $(i = 1, 2, \ldots)$ the vector fields and the one-forms obtained by the iterated action of N and N^*.

If there exist $(n-1)$ independent functions H_i $(i = 2, \ldots, n)$ and $(n(n+1)/2 - 1)$ functions ρ_{ij} $(i = 2, \ldots, n; 1 \leqslant j \leqslant i)$ with $\rho_{11} = 1$ and $\rho_{ii} \neq 0$ $(i = 2, \ldots, n)$, such that the one-forms α_i can be written as $\alpha_i = \sum_{j=1}^{i} \rho_{ij}\,\mathrm{d}H_j$ $(i = 1, 2, \ldots, n)$, then:

(i) the vector fields X_i satisfy the recursion relations $X_{i+1} = P_0\,\alpha_{i+1} = P_1\,\alpha_i$ $(i = 1, \ldots, n-1)$.

(ii) The functions H_i are in involution with respect to the Poisson bracket defined by P_0 and they are constants of motion for each field X_k $(k = 1, \ldots, n)$.

(iii) The Hamiltonian system corresponding to the vector field, X, is Liouville-integrable. Moreover, if P_1 is a Poisson tensor, then also X_2 is an integrable Hamiltonian vector field and the functions H_i are in involution also with respect to the Poisson bracket defined by P_1.

This result is applied in the next section of this paper, where we consider two Hénon–Heiles-type systems with three and four degrees of freedom.

To fix the notations, on any open set of a $2n$-dimensional symplectic manifold M, let $(\boldsymbol{q} = (q_1, \ldots, q_n); \boldsymbol{p} = (p_1, \ldots, p_n))$ be a set of canonical coordinates and P_0 the Poisson tensor $P_0 = \begin{bmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{bmatrix}$ ($\boldsymbol{I}$ denoting the $n \times n$ identity matrix). Let P_1 be a compatible Poisson tensor with respect to P_0, such that the Nijenhuis tensor $N := P_1 P_0^{-1}$ is maximal, i.e. it has n distinct eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. As is known [8], in a neighbourhood of a regular point, where the eigenvalues $\boldsymbol{\lambda}$ are independent, one can construct a canonical transformation $(\boldsymbol{q}; \boldsymbol{p}) \mapsto (\boldsymbol{\lambda}; \boldsymbol{\mu})$ ($(\boldsymbol{\lambda}; \boldsymbol{\mu})$ referred to as Nijenhuis coordinates) such that P_1 and N take the Darboux form

$$P_1 = \begin{bmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{bmatrix} \qquad N = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \qquad (\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n). \tag{1.3}$$

A QBH vector field is said to be Pfaffian [7] if the integrating factor ρ in equation (1.2) is the product of the eigenvalues of N, i.e.

$$\rho = \Pi_{i=1}^{n} \lambda_i. \tag{1.4}$$

Working in this setting,

- in section 2 we present two Hénon–Heiles-type systems with three and four degrees of freedom, which are Pfaffian QBH systems; passing to a set of Nijenhuis coordinates, we show that the Hamilton–Jacobi equations for these systems are separable;
- in section 3 we obtain the general solution to equation (1.2) for a Pfaffian QBH vector field with an arbitrary number of degrees of freedom; the Hamiltonian H and the function F contain n arbitrary smooth functions f_i, each one of them depending on a single pair $(\lambda_i; \mu_i)$ of Nijenhuis coordinates. Finally, we prove that the Hamiltonian H is separable.

2. Two Hénon–Heiles-type systems with three and four degrees of freedom

In this section we present two separable QBH systems with three and four degrees of freedom; they belong to a family of integrable flows obtained in [4] as stationary flows of the Korteweg–de Vries hierarchy [9]. This family contains the classical Hénon–Heiles system as its second member, so the higher members can be considered as multi-dimensional extensions of Hénon–Heiles.

The third member of this family, which is a stationary reduction of the seventh-order KdV flow, is defined in a six-dimensional phase space (with coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$, $\boldsymbol{p} = (p_1, p_2, p_3)$) by the Hamiltonian vector field $X = P_0\,\mathrm{d}H$, with the Hamiltonian function

$$H = \frac{1}{2}(2p_1p_2 + p_3^2) - \frac{5}{8}q_1^4 + \frac{5}{2}q_1^2q_2 + \frac{q_1q_3^2}{2} - \frac{q_2^2}{2}. \tag{2.1}$$

First, we can show that the vector field X is Liouville-integrable. Indeed, if one introduces the functions

$$\begin{aligned} H_1 &= H \\ H_2 &= \frac{p_1^2}{2} + p_1p_2q_1 + p_3^2q_1 - p_2^2q_2 - p_2p_3q_3 - \frac{q_1^5}{2} - \frac{q_1^2q_3^2}{4} + \frac{q_2q_3^2}{2} + 2q_1q_2^2 \\ H_3 &= \frac{p_3^2q_1^2}{2} + p_3^2q_2 - p_1p_3q_3 - p_2p_3q_1q_3 + \frac{p_2^2q_3^2}{2} + \frac{q_1^3q_3^2}{2} - q_1q_2q_3^2 - \frac{q_3^4}{8} \end{aligned} \tag{2.2}$$

X satisfies the assumptions of proposition 1.1; the tensor P_1 is given by

$$P_1 = \begin{bmatrix} 0 & A \\ -A^T & B \end{bmatrix} \qquad \mathbf{A} = -\begin{bmatrix} q_1 & -1 & 0 \\ 2q_2 & q_1 & q_3 \\ q_3 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & -p_2 & -p_3 \\ p_2 & 0 & 0 \\ p_3 & 0 & 0 \end{bmatrix} \tag{2.3}$$

and the functions ρ_{ij} are: $\rho_{11} = \rho_{22} = \rho_{33} = 1$, $\rho_{21} = \rho_{32} = -2q_1$, $\rho_{31} = (3q_1^2 - 2q_2)$.

Furthermore, one easily verifies that P_1 is a Poisson tensor, compatible with P_0 (so that $N = P_1P_0^{-1}$ is a Nijenhuis tensor). One can show that X is a QBH vector field; in fact equation (1.2) is verified with ρ and F given by $\rho = q_3^2$ and $F = H_3$.

Finally, let us show the separability of this system in terms of Nijenhuis coordinates. In this case the construction of a canonical map $\Phi : (\boldsymbol{\lambda}; \boldsymbol{\mu}) \mapsto (\boldsymbol{q}; \boldsymbol{p})$ between a set of Nijenhuis coordinates $(\boldsymbol{\lambda}; \boldsymbol{\mu})$ and the coordinates $(\boldsymbol{q}; \boldsymbol{p})$ is quite simple. We observe that the matrix $\mathbf{A}$ in equation (2.3) depends only on the coordinates $\boldsymbol{q}$, so also the eigenvalues $\boldsymbol{\lambda}$ depend only on $\boldsymbol{q}$: $q_k = f_k(\boldsymbol{\lambda})$. Then we introduce the generating function $S = \sum_{k=1}^{3} p_k f_k(\boldsymbol{\lambda})$ and we get

$$
\begin{aligned}
q_1 &= -\tfrac{1}{2}(\lambda_1+\lambda_2+\lambda_3)\\
q_2 &= -\tfrac{1}{8}(\lambda_1+\lambda_2+\lambda_3)^2+\tfrac{1}{2}(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\\
q_3 &= (\lambda_1\lambda_2\lambda_3)^{1/2}\\
p_1 &= \frac{\lambda_1\mu_1}{\lambda_{12}\lambda_{13}}(-\lambda_1+\lambda_2+\lambda_3)+\frac{\lambda_2\mu_2}{\lambda_{21}\lambda_{23}}(\lambda_1-\lambda_2+\lambda_3)+\frac{\lambda_3\mu_3}{\lambda_{31}\lambda_{32}}(\lambda_1+\lambda_2-\lambda_3)\\
p_2 &= -2\left(\frac{\lambda_1\mu_1}{\lambda_{12}\lambda_{13}}+\frac{\lambda_2\mu_2}{\lambda_{21}\lambda_{23}}+\frac{\lambda_3\mu_3}{\lambda_{31}\lambda_{32}}\right)\\
p_3 &= 2(\lambda_1\lambda_2\lambda_3)^{1/2}\left(\frac{\mu_1}{\lambda_{12}\lambda_{13}}+\frac{\mu_2}{\lambda_{21}\lambda_{23}}+\frac{\mu_3}{\lambda_{31}\lambda_{32}}\right)
\end{aligned}
\tag{2.4}
$$

where we put, for brevity, $\lambda_{ij} := \lambda_i - \lambda_j$. Since $\rho = q_3^2 = \lambda_1\lambda_2\lambda_3$, we are faced with a Pfaffian system. Written in the above-mentioned Nijenhuis coordinates, the Hamiltonian function, H, given by equation (2.1) takes the form

$$
H = \frac{\lambda_1(16\mu_1^2-\lambda_1^5)}{8\lambda_{12}\lambda_{13}}+\frac{\lambda_2(16\mu_2^2-\lambda_2^5)}{8\lambda_{21}\lambda_{23}}+\frac{\lambda_3(16\mu_3^2-\lambda_3^5)}{8\lambda_{31}\lambda_{32}}. \tag{2.5}
$$

It is easy to show that the Hamilton–Jacobi equation $H(\boldsymbol{\lambda}, \frac{\partial W}{\partial \boldsymbol{\lambda}}) = h$ is separable and has the complete integral $W = \sum_{i=1}^{3} W_i(\lambda_i; c_0, c_1, c_2)$, with W_1, W_2 and W_3 solutions of the following equations

$$
\frac{\mathrm{d}W_i}{\mathrm{d}\lambda_i} = \left(\frac{1}{16\lambda_i}(\lambda_i^6+c_2\lambda_i^2+c_1\lambda_i+c_0)\right)^{1/2} \qquad c_2 = 8h \qquad (i=1,2,3). \tag{2.6}
$$

Our second example is a Hénon–Heiles system with four degrees of freedom. It can be constructed as a stationary reduction of the ninth-order KdV flow [10]. Its phase space is eight dimensional, and the Hamiltonian is

$$
H = \frac{1}{2}(p_4^2+2p_1p_3+p_2^2)+\frac{3}{4}q_1^5-\frac{5}{2}q_1^3q_2+2q_1q_2^2+\frac{5}{2}q_1^2q_3+\frac{q_1q_4^2}{2}-q_2q_3. \tag{2.7}
$$

Also in this case, the vector field $X = P_0\,\mathrm{d}H$ is Liouville-integrable. Indeed, let us consider the functions

$$
\begin{aligned}
H_1 &= H\\
H_2 &= p_1p_2+p_2^2q_1+p_1p_3q_1+p_4^2q_1-p_2p_3q_2-p_3^2q_3-p_3p_4q_4\\
&\quad +\frac{5}{8}q_1^6-\frac{5}{4}q_1^4q_2-q_1^2q_2^2-\frac{q_1^2q_4^2}{4}+q_2^3+\frac{q_2q_4^2}{2}+3q_1q_2q_3-\frac{1}{2}q_3^2\\
H_3 &= \tfrac{1}{2}p_2^2q_1^2+\tfrac{1}{2}p_4^2q_1^2+\tfrac{1}{2}p_3^2q_2^2+p_2p_3q_1q_2+p_3^2q_4^2-p_3p_4q_1q_4\\
&\quad -2p_2p_3q_3+p_4^2q_2+p_1p_3q_2+p_1p_2q_1-p_2p_4q_4+\tfrac{1}{2}p_1^2\\
&\quad +\tfrac{5}{4}q_1^5q_2-3q_1^3q_2^2+\tfrac{1}{2}q_1^3q_4^2+\tfrac{5}{4}q_1^4q_3+q_1q_2^3-q_1^2q_2q_3-\tfrac{1}{2}q_1q_2q_4^2\\
&\quad +\tfrac{1}{2}q_3q_4^2+q_2^2q_3+2q_1q_3^2\\
H_4 &= -p_2p_4q_1q_4-p_3p_4q_2q_4+p_2p_3q_4^2+p_4^2q_1q_2+p_4^2q_3-p_1p_4q_4\\
&\quad -\tfrac{5}{8}q_1^4q_4^2+\tfrac{3}{2}q_1^2q_2q_4^2-\tfrac{1}{2}q_2^2q_4^2-q_1q_3q_4^2-\tfrac{1}{8}q_4^4
\end{aligned}
\tag{2.8}
$$

and the tensor $P_1 = \begin{bmatrix} 0 & A \\ -A^T & B \end{bmatrix}$, with the matrices **A** and **B** given by

$$\mathbf{A} = -\begin{bmatrix} q_1 & -1 & 0 & 0 \\ q_2 & 0 & -1 & 0 \\ 2q_3 & q_2 & q_1 & q_4 \\ q_4 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & -p_2 & -p_3 & -p_4 \\ p_2 & 0 & 0 & 0 \\ p_3 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 0 \end{bmatrix}. \tag{2.9}$$

Then X verifies the assumptions of proposition 1.1 with the following choices for the functions ρ_{ij}: $\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44} = 1$, $\rho_{21} = \rho_{32} = \rho_{43} = -2q_1$, $\rho_{31} = \rho_{42} = (3q_1^2 - 2q_2)$, $\rho_{41} = (-4q_1^3 + 6q_1q_2 - 2q_3)$.

Moreover, P_1 is a Poisson tensor, compatible with P_0 (so that $N = P_1 P_0^{-1}$ is a Nijenhuis tensor). The Hamiltonian vector field X is a QBH vector field since it satisfies the equation $X = P_1\,\mathrm{d}F/\rho$, with $\rho = -q_4^2$, $F = -H_4$.

Finally, let us consider the map between the coordinates $(\boldsymbol{q}; \boldsymbol{p})$ and the Nijenhuis coordinates $(\boldsymbol{\lambda}; \boldsymbol{\mu})$. Since also in this case the matrix **A** in equation (2.9) depends only on $\boldsymbol{q}$, we proceed as in the previous example. The result is

$$\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -2q_1 \\
&\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = q_1^2 + 2q_2 \\
&\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4 = -2(q_1q_2 + q_3) \\
&\lambda_1\lambda_2\lambda_3\lambda_4 = -q_4^2 \\
&\mu_1 = -\frac{p_1}{2} - \frac{p_4}{2}\frac{\lambda_2\lambda_3\lambda_4}{(-\lambda_1\lambda_2\lambda_3\lambda_4)^{1/2}} + \frac{p_2}{4}(-\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \\
&\qquad + \frac{p_3}{16}(-3\lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2 + 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_3^2 \\
&\qquad + 2\lambda_1\lambda_4 - 2\lambda_2\lambda_4 - 2\lambda_3\lambda_4 + \lambda_4^2) \\
&\mu_2 = -\frac{p_1}{2} - \frac{p_4}{2}\frac{\lambda_1\lambda_3\lambda_4}{(-\lambda_1\lambda_2\lambda_3\lambda_4)^{1/2}} + \frac{p_2}{4}(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4) \\
&\qquad + \frac{p_3}{16}(\lambda_1^2 + 2\lambda_1\lambda_2 - 3\lambda_2^2 - 2\lambda_1\lambda_3 + 2\lambda_2\lambda_3 \\
&\qquad + \lambda_3^2 - 2\lambda_1\lambda_4 + 2\lambda_2\lambda_4 - 2\lambda_3\lambda_4 + \lambda_4^2) \\
&\mu_3 = -\frac{p_1}{2} - \frac{p_4}{2}\frac{\lambda_1\lambda_2\lambda_4}{(-\lambda_1\lambda_2\lambda_3\lambda_4)^{1/2}} + \frac{p_2}{4}(\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4) \\
&\qquad + \frac{p_3}{16}(\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 + 2\lambda_1\lambda_3 + 2\lambda_2\lambda_3 - 3\lambda_3^2 \\
&\qquad - 2\lambda_1\lambda_4 - 2\lambda_2\lambda_4 + 2\lambda_3\lambda_4 + \lambda_4^2) \\
&\mu_4 = -\frac{p_1}{2} - \frac{p_4}{2}\frac{\lambda_1\lambda_2\lambda_3}{(-\lambda_1\lambda_2\lambda_3\lambda_4)^{1/2}} + \frac{p_2}{4}(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4) \\
&\qquad + \frac{p_3}{16}(\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_3^2 \\
&\qquad + 2\lambda_1\lambda_4 + 2\lambda_2\lambda_4 + 2\lambda_3\lambda_4 - 3\lambda_4^2).
\end{aligned} \tag{2.10}$$

By solving this system with respect to $(\boldsymbol{q}; \boldsymbol{p})$ one can recover the canonical map $\Phi : (\boldsymbol{\lambda}; \boldsymbol{\mu}) \mapsto (\boldsymbol{q}; \boldsymbol{p})$ which allows one to write the Hamiltonian function H given by equation (2.7) in terms of Nijenhuis coordinates; it reads

$$H = \frac{\lambda_1(16\mu_1^2 - \lambda_1^7)}{8\lambda_{12}\lambda_{13}\lambda_{14}} + \frac{\lambda_2(16\mu_2^2 - \lambda_2^7)}{8\lambda_{21}\lambda_{23}\lambda_{24}} + \frac{\lambda_3(16\mu_3^2 - \lambda_3^7)}{8\lambda_{31}\lambda_{32}\lambda_{34}} + \frac{\lambda_4(16\mu_4^2 - \lambda_4^7)}{8\lambda_{41}\lambda_{42}\lambda_{43}}. \tag{2.11}$$

Let us remark that also in this case the system is Pfaffian, since $\rho = -q_4^2 = \lambda_1\lambda_2\lambda_3\lambda_4$. Finally, one proves that the Hamilton–Jacobi equation $H(\boldsymbol{\lambda}; \frac{\partial W}{\partial \boldsymbol{\lambda}}) = h$ is separable and has the complete integral $W = \sum_{i=1}^{4} W_i(\lambda_i; c_0, c_1, c_2, c_3)$, with W_1, W_2, W_3 and W_4 solutions of the following equations

$$\frac{\mathrm{d}W_i}{\mathrm{d}\lambda_i} = \left(\frac{1}{16\lambda_i}(\lambda_i^8 + c_3\lambda_i^3 + c_2\lambda_i^2 + c_1\lambda_i + c_0\right)^{1/2} \qquad c_3 = 8h \qquad (i = 1, 2, 3, 4). \tag{2.12}$$

3. Quasi-bi-Hamiltonian systems with n degrees of freedom

Let us consider a $2n$-dimensional symplectic manifold M, a Poisson tensor P_1 compatible with P_0, and let us assume to have introduced a set of Nijenhuis coordinates $(\boldsymbol{\lambda}; \boldsymbol{\mu})$, so that P_1 takes the Darboux form (1.3). We search for the general solution of the QBH equation (1.2) in the Pfaffian case (i.e. with ρ defined by equation (1.4)).

Proposition 3.1. In the Pfaffian case, the general solution of the equation $P_0\,\mathrm{d}H = P_1\,\mathrm{d}F/\rho$ is given by

$$H = \sum_{i=1}^{n} \frac{1}{\Delta_i} f_i(\lambda_i; \mu_i) \qquad F = \sum_{i=1}^{n} \frac{\rho_i}{\Delta_i} f_i(\lambda_i; \mu_i) \tag{3.1}$$

where $(\boldsymbol{\lambda}; \boldsymbol{\mu})$ are Nijenhuis coordinates, $\Delta_i := \Pi_{j\neq i}\lambda_{ij}$ $(\lambda_{ij} := \lambda_i - \lambda_j)$, $\rho_i := \rho/\lambda_i$ and the n functions $f_i(\lambda_i; \mu_i)$ (each one of them depending on one pair of coordinates) are arbitrary smooth functions.

Proof. Equation (1.2) corresponds to the two sets of equations

$$\frac{\partial H}{\partial \mu_i} = \frac{\lambda_i}{\rho}\frac{\partial F}{\partial \mu_i} \qquad (i = 1, 2, \ldots, n) \tag{3.2}$$

$$\frac{\partial H}{\partial \lambda_i} = \frac{\lambda_i}{\rho}\frac{\partial F}{\partial \lambda_i} \qquad (i = 1, 2, \ldots, n). \tag{3.3}$$

The general solution of the first set is

$$H = \frac{1}{\rho}\sum_{i=1}^{n} \lambda_i G_i(\boldsymbol{\lambda}; \mu_i) + K(\boldsymbol{\lambda}) \qquad F = \sum_{i=1}^{n} G_i(\boldsymbol{\lambda}; \mu_i) \tag{3.4}$$

where the functions $G_i = G_i(\boldsymbol{\lambda}; \mu_i)$ and $K = K(\boldsymbol{\lambda})$ are arbitrary. Indeed, the solution to the first equation (3.2), for $i = 1$, is $H = \frac{\lambda_1}{\rho}F(\boldsymbol{\lambda}; \boldsymbol{\mu}) + \phi_1(\boldsymbol{\lambda}; \mu_2, \ldots, \mu_n)$, with ϕ_1 arbitrary; on account of this result, equation (3.2) for $i = 2$ has the solution

$$\begin{aligned} H &= \frac{\lambda_1}{\rho}G_1(\boldsymbol{\lambda}; \mu_1) + \frac{\lambda_2}{\rho}\psi_1(\boldsymbol{\lambda}; \mu_2, \ldots, \mu_n) + \phi_2(\boldsymbol{\lambda}; \mu_3, \ldots, \mu_n) \\ F &= G_1(\boldsymbol{\lambda}; \mu_1) + \psi_1(\boldsymbol{\lambda}; \mu_2, \ldots, \mu_n) \end{aligned} \tag{3.5}$$

with ψ_1 and ϕ_2 arbitrary. Iterating this procedure for $i = 3, \ldots, n$ one easily obtains solution (3.4). Let us insert this solution into equation (3.3), putting into evidence the dependence on $\boldsymbol{\mu}$; we conclude that $K(\boldsymbol{\lambda})$ has to be a constant function (which can be taken as being equal to zero without loss of generality) and that equation (3.3) can be written as

$$\frac{\partial}{\partial \lambda_i}\left(\sum_{j=1}^{n} \lambda_{ij} G_j\right) = \frac{1}{\lambda_i}\left(\sum_{j=1}^{n} \lambda_{ij} G_j\right) \qquad (i = 1, 2, \ldots, n). \tag{3.6}$$

By integrating these equations ($i = 1, 2, \ldots, n$) and taking into account the dependence on $\boldsymbol{\mu}$, we easily obtain that

$$G_i(\boldsymbol{\lambda}; \boldsymbol{\mu}) = \frac{\rho_i}{\Delta_i} f_i(\lambda_i; \mu_i) \qquad (i = 1, 2, \ldots, n) \tag{3.7}$$

where each f_i is an arbitrary function depending only on the pair of variables $(\lambda_i; \mu_i)$. $\square$

Of course, the vector field $X = P_0\, \mathrm{d}H$ is a QBH vector field in $2n$ dimensions.

Finally, from the above result, we can also prove that the Hamiltonian H and the function F are separable.

Proposition 3.2. The Hamiltonian H and the function F, written in terms of the Nijenhuis coordinates $(\boldsymbol{\lambda}; \boldsymbol{\mu})$ in the form of (3.1), are separable for each n-ple of functions $f_i(\lambda_i; \mu_i)$.

Proof. The Hamilton–Jacobi equation for H is separable iff H verifies the Levi-Civita conditions $L_{ij}(H) = 0$ $(i, j = 1, \ldots, n;\ i \neq j)$ where [11]

$$L_{ij}(H) = \frac{\partial H}{\partial \lambda_i}\frac{\partial H}{\partial \lambda_j}\frac{\partial^2 H}{\partial \mu_i \partial \mu_j} + \frac{\partial H}{\partial \mu_i}\frac{\partial H}{\partial \mu_j}\frac{\partial^2 H}{\partial \lambda_i \partial \lambda_j} - \frac{\partial H}{\partial \lambda_i}\frac{\partial H}{\partial \mu_j}\frac{\partial^2 H}{\partial \mu_i \partial \lambda_j} - \frac{\partial H}{\partial \lambda_j}\frac{\partial H}{\partial \mu_i}\frac{\partial^2 H}{\partial \mu_j \partial \lambda_i}. \tag{3.8}$$

In our case, it is $\partial^2 H/\partial \mu_i \partial \mu_j = 0$ and

$$\frac{\partial \Delta_j}{\partial \lambda_j} = \Delta_j \sum_{\alpha \neq j} \lambda_{j\alpha}^{-1} \qquad \frac{\partial \Delta_j}{\partial \lambda_\beta} = -\Delta_j \lambda_{j\beta}^{-1} \qquad (\beta \neq j). \tag{3.9}$$

It may be useful to decompose $L_{ij}(H)$ as $L_{ij}(H) = M_{ij}(H) + N_{ij}(H)$, where $M_{ij}(H)$ depends linearly on the functions f_i, and $N_{ij}(H)$ depends on the derivatives $\partial f_i/\partial \lambda_i$ but not on f_i. By using equation (3.9) one can directly verify that $M_{ij}(H) = 0$ and $N_{ij}(H) = 0$. Similarly, one can show that the Levi-Civita conditions (3.8) are fulfilled also by the function F given in (3.1). $\square$

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